Homework #9 Solutions

Section 8.1

2. $a_n = \dfrac{1}{n!}$

$a_1 = \dfrac{1}{1!} = 1, \quad a_2 = \dfrac{1}{2!} = \dfrac{1}{2}, \quad a_3 = \dfrac{1}{3!} = \dfrac{1}{6}, \quad a_4 = \dfrac{1}{4!} = \dfrac{1}{24}$

16. Sequence $1, -\frac{1}{4}, \frac{1}{9}, \frac{-1}{16}, \frac{1}{25}, \ldots$

$$\boxed{a_n = (-1)^{n+1}\frac{1}{n^2}}$$

24. $a_n = \dfrac{n+(-1)^n}{n}$.

The sequence $\boxed{\text{converges to } 1}$ because $\displaystyle\lim_{n\to\infty} \frac{n+(-1)^n}{n} = \lim_{n\to\infty} \frac{n \pm 1}{n} = 1$.

27. $a_n = \dfrac{1-5n^4}{n^4+8n^3}$

The sequence $\boxed{\text{converges to } -5}$ because $\displaystyle\lim_{n\to\infty} \frac{1-5n^4}{n^4+8n^3} = \lim_{n\to\infty} \frac{\frac{1}{n^4}-5}{1+\frac{8}{n}} = -5$.

40. $a_n = n\pi \cos(n\pi)$

The sequence $\boxed{\text{diverges}}$: $a_1 = -\pi, \quad a_2 = 2\pi, \quad a_3 = -3\pi, \ldots$

So $a_n = (-1)^n n\pi$. As $n \to \infty$, $|a_n| \to \infty$.

52. $a_n = \sqrt[n]{n^2}$.

The sequence $\boxed{\text{converges to } 1}$ because:

$$\lim_{n\to\infty} n^{\frac{2}{n}} = \lim_{n\to\infty} e^{\ln(n^{\frac{2}{n}})} = \lim_{n\to\infty} e^{\frac{2}{n}\ln n}$$

But $\displaystyle\lim_{n\to\infty} \frac{2\ln n}{n} = \lim_{n\to\infty} \frac{2\cdot\frac{1}{n}}{1}$ (L'Hospital)

$= 0$.

So $\displaystyle\lim_{n\to\infty} n^{\frac{2}{n}} = \lim_{n\to\infty} e^{\frac{2\ln n}{n}} = e^0 = 1$

63. $a_n = \left(\frac{1}{n}\right)^{\frac{1}{\ln n}}$.

Write $a_n = e^{\ln\left(\left(\frac{1}{n}\right)^{\frac{1}{\ln n}}\right)} = e^{\left(\frac{\ln(\frac{1}{n})}{\ln n}\right)} = e^{\left(\frac{\ln(n^{-1})}{\ln n}\right)} = e^{-\frac{\ln n}{\ln n}} = e^{-1}$.

So $a_n = \frac{1}{e}$ for all n, and $\lim_{n\to\infty} a_n = \frac{1}{e}$.

So the sequence $\boxed{\text{converges to } \frac{1}{e}}$

100. $a_n = 2 - \frac{2}{n} - \frac{1}{2^n}$.

$\{a_n\}$ is bounded above by 2 since $a_n \leq 2$ for all n.

The sequence is nondecreasing because:

$$\frac{2}{n} \geq \frac{2}{n+1} \quad \text{and} \quad \frac{1}{2^n} \geq \frac{1}{2^{n+1}}$$

So $$2 - \frac{2}{n} - \frac{1}{2^n} \leq 2 - \frac{2}{n+1} - \frac{1}{2^{n+1}}.$$

In other words, $a_n \leq a_{n+1}$.

Section 8.2

2. $\frac{9}{100} + \frac{9}{100^2} + \frac{9}{100^3} + \dots + \frac{9}{100^n} + \dots$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{\frac{9}{100}\left(1-\frac{1}{100^n}\right)}{1-\frac{1}{100}} = \frac{\cancel{9}}{\cancel{100}}\left(1-\frac{1}{100^n}\right)\cdot\frac{\cancel{100}}{\cancel{99}_{11}}$$

$$= \boxed{\frac{1-\frac{1}{100^n}}{11}}$$

The series converges to $\boxed{\frac{1}{11}}$

6. $\frac{5}{1\cdot 2} + \frac{5}{2\cdot 3} + \frac{5}{3\cdot 4} + \dots + \frac{5}{n(n+1)} + \dots$

$\frac{5}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$ (Partial Fraction decomposition).

$A(n+1) + B(n) = 5 \longrightarrow A = 5,\ B = -5$ (from plugging in $n=0$, $n=-1$).

So our series becomes:

$$\left(\frac{5}{1} - \frac{5}{2}\right) + \left(\frac{5}{2} - \frac{5}{3}\right) + \left(\frac{5}{3} - \frac{5}{4}\right) + \dots + \left(\frac{5}{n} - \frac{5}{n+1}\right) + \dots$$

and $\boxed{S_n = 5 - \frac{5}{n+1}}$. The series converges to $\boxed{5}$.

8. $\sum_{n=2}^{\infty} \frac{1}{4^n} = \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots$

The series is geometric with $a = \frac{1}{16}$, $r = \frac{1}{4}$.

So the series converges to $\frac{\frac{1}{16}}{1-\frac{1}{4}} = \frac{1}{16} \cdot \frac{4}{3} = \boxed{\frac{1}{12}}$

14. $\sum_{n=0}^{\infty} \left(\frac{2^{n+1}}{5^n}\right) = 2 + \frac{4}{5} + \frac{8}{25} + \cdots$

The series is geometric with $a = 2$, $r = \frac{2}{5}$.

So the series converges to $\frac{2}{1-\frac{2}{5}} = \frac{2 \cdot 5}{3} = \boxed{\frac{10}{3}}$

19. $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = \left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right) + \cdots$

So $S_n = 1 - \frac{1}{\sqrt{n+1}}$, and the series converges to $\boxed{1}$

24. $\sum_{n=0}^{\infty} (\sqrt{2})^n$. This is a geometric series with $a = 1$, $r = \sqrt{2}$.

Since $|\sqrt{2}| > 1$, the series $\boxed{\text{diverges}}$

27. $\sum_{n=0}^{\infty} \cos n\pi = 1 - 1 + 1 - 1 + 1 - 1 \ldots$

The partial sums of this series are $1, 0, 1, 0, \ldots$

Since the sequence of partial sums does not converge, this series $\boxed{\text{diverges}}$

28. $\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n} = 1 - \frac{1}{5} + \frac{1}{25} - \frac{1}{125} + \cdots = \sum_{n=0}^{\infty} \left(\frac{-1}{5}\right)^n$.

This is a geometric series with $a = 1$, $r = \frac{-1}{5}$.

Since $|r| = \frac{1}{5} < 1$, the series $\boxed{\text{converges}}$ to $\frac{1}{1-\frac{-1}{5}} = \boxed{\frac{5}{6}}$